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LETTER TO THE EDITOR

Expansion coefficient of heat kernel of Laplacian operator in Riemann–Cartan space

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Abstract. The method of determining the coefficient in the asymptotic expansion of the heat kernel of the Laplacian operator in a Riemann–Cartan space is discussed. In the context of SO(4) gravity with totally antisymmetric torsion tensor, the method of coincidence limits of De Witt is equivalent to that of the algorithm of 't Hooft. As an example, the axial current divergence due to the spin-$\frac{1}{2}$ field is given.

Recently Goldthorpe (1980) attempted to derive the formula for the so-called b_4 coefficients in the asymptotic expansion of the trace of the heat kernel of a second-order Laplacian type operator Δ in a Riemann–Cartan space U_4. He started from the equation which is a U_4 extension of the equation of motion for an irreducible spin-$(A+B)$ field $\phi(A, B)$ in the SO(4) gravity in the Riemannian space (Christensen and Duff 1979)

$$\Delta\phi(A, B) \equiv (-g^{\mu\nu}D_\mu D_\nu + S^\mu D_\mu + X)\phi(A, B) = 0, \tag{1}$$

$\phi(A, B)$ transforming as a scalar under the action of the coordinate transformation. D_μ is defined for a vector spinor as

$$D_\mu\phi_\nu = (\partial_\mu + B_\mu)\phi_\nu - \Gamma_{\nu}{}^{\lambda}{}_{\mu}\phi_\lambda, \tag{2}$$

where $B_\mu = B^{ab}_\mu\, \Sigma_{ab}$ in which Σ_{ab} is the SO(4) generator in the $(A.B)$ representation. The X and S^μ are

$$\begin{aligned}
&X = -\Sigma^{\mu\nu}Y_{\mu\nu} = -\Sigma^{\mu\nu}F^{ab}{}_{\mu\nu}\Sigma_{ab},\\
&S^\mu = -2\Sigma^{\lambda\rho}S_{\lambda\rho}{}^{\mu}, \qquad \text{for } A+B = \text{integer},\\
&X = -(1/A)\Sigma^{\mu\nu}Y_{\mu\nu}{}^{(+)} = -(1/A)\Sigma^{\mu\nu}F^{ab}_{(+)\mu\nu}\Sigma_{ab},\\
&S^\mu = -(2/A)\Sigma^{\lambda\rho}S_{\lambda\rho}{}^{\mu}, \qquad \text{for } A+B = \text{half-integer}, A > B,\\
&X = -(1/B)\Sigma^{\mu\nu}Y_{\mu\nu}{}^{(-)} = -(1/B)\Sigma^{\mu\nu}F^{ab}_{(-)\mu\nu}\Sigma_{ab},\\
&S^\mu = -(2/B)\Sigma^{\lambda\rho}S_{\lambda\rho}{}^{\mu}, \qquad \text{for } A+B = \text{half-integer}, A < B,
\end{aligned} \tag{3}$$

in which

$$Y^{(\pm)}_{\mu\nu} = \tfrac{1}{2}(Y_{\mu\nu} \pm \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho}Y^{\lambda\rho}), \tag{4}$$

where

$$Y_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = F^{ab}{}_{\mu\nu}\Sigma_{ab} \tag{5}$$

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and

$$S_{\mu\nu\lambda} = g_{\lambda\rho} S_{\mu\nu}{}^{\rho} = \tfrac{1}{2} g_{\lambda\rho} (\Gamma_{\mu}{}^{\rho}{}_{\nu} - \Gamma_{\nu}{}^{\rho}{}_{\mu}). \tag{6}$$

Goldthorpe determined the b_4 coefficients with the aid of the method of coincidence limits of De Witt (1965). In U_4, the essential feature different from the Riemannian case is in the definition of the autoparallel displacement matrix $I(x, x')$ for the field

$$\sigma_{;}{}^{\mu}(D_{\mu} - \tfrac{1}{2}S_{\mu})I = 0, \tag{7}$$

where the bi-scalar $\sigma(x, x')$ is one half of the square of the distance along an autoparallel from the point x to x', and the semi-colon denotes the covariant differentiation (with respect to $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$). The equation (7) is introduced to ensure the condition $\lim_{x' \to x} I(x, x') =$ unit matrix. By making the restriction that the torsion be totally antisymmetric, Goldthorpe obtained the b_4 coefficient with 49 terms tabulated in his tables 1 and 2.

We note here that the coefficient b_4 is directly related to the counter Lagrangian ΔL to eliminate one-loop divergence:

$$\Delta L = (1/\varepsilon) g^{1/2} (-1)^{2(A+B)} \tfrac{1}{2} \{ b_4(A, B) + b_4(B, A) \}, \tag{8}$$

where $\varepsilon = 16\pi^2(n-4)$. Since a scalar field ($A = B = 0$) does not couple with the torsion, the coefficient b_4 for the scalar field should not contain $S_{\mu\nu\lambda}$. In other words, in Goldthorpe's table 1, the sum of terms without the generators Σ_{ab} should coincide with that of the Riemannian case. It seems that this is not the case. I suppose that there are mistakes in the coefficients of the terms of table 1. Taking account of this fact and rewriting the terms in table 2, we have the formula

$$\begin{aligned} b_4 = \mathrm{Tr}[& \tfrac{1}{180} R_{\mu\nu\lambda\rho}(\{\ \}) R^{\mu\nu\lambda\rho}(\{\ \}) - \tfrac{1}{180} R_{\mu\nu}(\{\ \}) R^{\mu\nu}(\{\ \}) \\ & - \tfrac{1}{6} X_{|\mu}{}^{\mu} + \tfrac{1}{30} R(\{\ \})_{|\mu}{}^{\mu} - \tfrac{1}{24}(S_{\mu}S^{\mu})_{|\nu}{}^{\nu} + \tfrac{1}{12}\{Y_{\mu\nu} + S_{\mu\nu}\} \\ & \times \{Y^{\mu\nu} + S^{\mu\nu}\} + \tfrac{1}{2}\{X - \tfrac{1}{2}\tilde{D}_{\mu}S^{\mu} + \tfrac{1}{4}S_{\mu}S^{\mu} - \tfrac{1}{6}R(\{\ \})\}^2], \end{aligned} \tag{9}$$

in which

$$S_{\mu\nu} = -\tfrac{1}{2}(\tilde{D}_{\mu}S_{\nu} - \tilde{D}_{\nu}S_{\mu}) + \tfrac{1}{4}(S_{\mu}S_{\nu} - S_{\nu}S_{\mu})$$

and the curvature tensors are defined in terms of the Christoffel symbol $\{{}^{\lambda}_{\mu\nu}\}$, the vertical line denotes the covariant differentiation with respect to $\{{}^{\lambda}_{\mu\nu}\}$ and $\tilde{D}_{\mu}$ is D_{μ} in which $\Gamma^{\lambda}_{\mu\nu}$ is replaced by $\{{}^{\lambda}_{\mu\nu}\}$.

It is also shown that the above formula is derived from the algorithm of 't Hooft and Veltman (1974) based on Feynman graph analysis. Doubling the number of fields and going over to a complex basis, we set up the Lagrangian

$$\begin{aligned} L = -g^{1/2}\{ & \phi^*(A, B) D_{\mu} g^{\mu\nu} D_{\nu} \phi(A, B) \\ & + \phi^*(A, B) S^{\mu} D_{\mu} \phi(A, B) + \phi^*(A, B) \times \phi(A, B)\}, \end{aligned} \tag{10}$$

which leads to (1). From this form and the 't Hooft algorithm we can immediately derive the formula (9) except for the terms which are written in the form of a total derivative. The latter terms are introduced on consulting the result of Gilkey (1975).

The calculation of the trace is carried out and the conformal and axial anomalies due to fields of arbitrary spin in U_4 are obtained. As an illustration, we have the axial current divergence due to a Majorana spin-$\frac{1}{2}$ field,

$$D^\mu J^5_\mu = g^{1/2}[b_4(\tfrac{1}{2}, 0) - b_4(0, \tfrac{1}{2})]$$
$$= (4\pi)^{-2} g^{1/2}[-\tfrac{1}{48}\varepsilon_{\mu\nu\lambda\rho} R^{\alpha\beta\mu\nu}(\{\ \}) R_{\alpha\beta}{}^{\lambda\rho}(\{\ \})$$
$$+ \tfrac{5}{24}\varepsilon^{\mu\nu\lambda\rho}(D_\mu A_\nu - D_\nu A_\mu)(D_\lambda A_\rho - D_\rho A_\lambda) + R^{\mu\nu}(\{\ \}) D_\mu A_\nu$$
$$- \tfrac{5}{12} R(\{\ \}) D_\mu A^\mu + 18 A^\mu A^\nu D_\mu A_\nu + \tfrac{81}{2} A_\rho^2 (D_\nu A^\mu) + D^\mu D_\mu (D_\rho A^\rho)], \quad (11)$$

where

$$A_\mu = -\tfrac{1}{6}\varepsilon_{\mu\nu\lambda\rho} S^{\nu\lambda\rho}.$$

When the torsion is zero ($A_\mu = 0$), equation (11) reduces to a well known result (Kimura 1969). It also shows that the right-hand side of (11) cannot be expressed in terms of the following topological invariant alone:

$$p = \frac{1}{32\pi^2} g^{1/2} \varepsilon_{\mu\nu\lambda\rho} F^{\alpha\beta\mu\nu} F_{\alpha\beta}{}^{\lambda\rho} = \frac{1}{32\pi^2} g^{1/2} \varepsilon_{\mu\nu\lambda\rho} R^{\alpha\beta\mu\nu} R_{\alpha\beta}{}^{\lambda\rho}$$
$$= \frac{1}{32\pi^2} g^{1/2}[\varepsilon_{\mu\nu\lambda\rho} R^{\alpha\beta\mu\nu}(\{\ \}) R_{\alpha\beta}{}^{\lambda\rho}(\{\ \}) + 2\varepsilon^{\mu\nu\lambda\rho}(D_\mu A_\nu - D_\nu A_\mu)$$
$$\times (D_\lambda A_\rho - D_\rho A_\lambda) + 16\{R_{\mu\nu}(\{\ \}) - \tfrac{1}{2} g_{\mu\nu} R(\{\ \})\} D^\mu A^\nu + 16 D^\mu (A_\mu A_\rho^2)]. \quad (12)$$

The detailed account will be reported elsewhere.

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